

### Definición de las sucesiones de paridad

Sean las sucesiones  $a_n = \frac{1-(-1)^n}{2}$  y  $b_n = \frac{n-a_n}{2}$ . Entonces  $b_n = \frac{2n-1+(-1)^n}{4}$ .

Por ejemplo, para  $x \in \mathbb{R}$ ,  $\frac{d^n}{dx^n}(\sin x) = (-1)^{b_n}(a_n \cos x + (1 - a_n) \sin x)$ . Ahora, haciendo uso de los polinomios de Taylor centrados en cero, se obtiene:

$$\sin x = \sum_{n=0}^{\infty} a_n (-1)^{b_n} \frac{x^n}{n!}$$

### Desarrollo de $e^{inx}$

Haciendo uso de las fórmulas de Euler, de De Moivre, y del Teorema del Binomio, se obtiene:

$$e^{inx} = \cos nx + i \sin nx = (\cos x + i \sin x)^n = \sum_{k=0}^n \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k \quad (0)$$

### Fórmulas de ángulo múltiple para el coseno usando sucesiones de paridad

De (0), se obtiene:

$$\begin{aligned} \cos nx &= \sum_{k=0}^{b_n} \binom{n}{2k} (-1)^k (\cos x)^{n-2k} (\sin x)^{2k} \\ &= \sum_{k=0}^{b_n} \binom{n}{2k} \left( \sum_{p=0}^k \binom{k}{p} (-1)^p (\cos x)^{n-2p} \right) \\ &= \sum_{k=0}^{b_n} \left( \sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \quad (1) \end{aligned}$$

Observación:

$$\sum_{k=0}^n u_k \left( \sum_{p=0}^k v_p w_{p,k} \right) = \sum_{k=0}^n \left( \sum_{p=k}^n u_p w_{p,k} \right) v_k$$

Luego,

$$\sum_{n=0}^m \cos nx = \sum_{n=0}^m \left( \sum_{k=0}^{b_{m-a_n}-b_n} \left( \sum_{p=k}^{b_{2k+n}} \binom{2k+n}{2p} \binom{p}{k} \right) (-1)^k \right) (\cos x)^n = \frac{\sin(m+1)\frac{x}{2}}{\sin \frac{x}{2}} \cos m \frac{x}{2} \quad (2)$$

Observación:

$$\sum_{n=0}^m \left( \sum_{k=0}^{b_n} u_k v_{n-2k} w_{n,k} \right) = \left[ \sum_{n=0}^{b_m} \left( \sum_{k=0}^{b_{m-n}} u_k w_{2k+2n,k} \right) v_{2n} \right] + \left[ \sum_{n=0}^{b_{m-1}} \left( \sum_{k=0}^{b_{m-1-n}} u_k w_{2k+(2n+1),k} \right) v_{2n+1} \right] = \sum_{n=0}^m \left( \sum_{k=0}^{b_{m-a_n}-b_n} u_k w_{2k+n,k} \right) v_n$$

Consideremos de (1):

$$\cos nx = (\cos x)^n \sum_{p=0}^{b_n} \binom{n}{2p} + \sum_{k=1}^{b_n} \left( \sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k}$$

Por consiguiente,

$$(\cos x)^n = \frac{1}{2^{n-1}} \left( \cos nx - \sum_{k=1}^{b_n} \left( \sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \right)$$

*Observación:*

$$\sum_{p=0}^{b_n} \binom{n}{2p} = 2^{n-1}$$

Ahora, usando (2) se obtiene:

$$\begin{aligned} \sum_{m=0}^q (\cos mx)^n &= \frac{1}{2^{n-1}} \left( \frac{\sin(q+1) \frac{nx}{2}}{\sin \frac{nx}{2}} \cos q \frac{nx}{2} - \sum_{m=0}^q \sum_{k=1}^{b_n} (-1)^k \left( \sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (\cos mx)^{n-2k} \right) \\ &= \frac{1}{2^{n-1}} \left( \frac{\sin(q+1) \frac{nx}{2}}{\sin \frac{nx}{2}} \cos q \frac{nx}{2} - \sum_{k=1}^{b_n} (-1)^k \left( \sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) \left( \sum_{m=0}^q (\cos mx)^{n-2k} \right) \right) \quad (*) \end{aligned}$$

*Observación:*

$$\sum_{m=0}^q \left( \sum_{k=1}^n u_k v_{m,k} \right) = \sum_{k=1}^n u_k \left( \sum_{m=0}^q v_{m,k} \right)$$

### **Fórmulas de ángulo múltiple para el seno usando sucesiones de paridad**

De (0), se obtiene:

$$\begin{aligned} \sin nx &= \sum_{k=0}^{b_{n-1}} (-1)^k \binom{n}{2k+1} (\cos x)^{n-(2k+1)} (\sin x)^{2k+1} \\ &= (\cos x)^{a_{n-1}} \sum_{k=0}^{b_{n-1}} \binom{n}{2k+1} \left( \sum_{p=k}^{b_{n-1}} (-1)^p \binom{b_{n-1}-k}{b_{n-1}-p} (\sin x)^{2p+1} \right) \\ &= (\cos x)^{a_{n-1}} \sum_{k=0}^{b_{n-1}} \left( \sum_{p=0}^k \binom{n}{2p+1} \binom{b_{n-1}-p}{b_{n-1}-k} \right) (-1)^k (\sin x)^{2k+1} \quad (3) \end{aligned}$$

*Observación:*

$$\sum_{k=0}^n u_k \left( \sum_{p=k}^n v_p w_{p,k} \right) = \sum_{k=0}^n \left( \sum_{p=0}^k u_p w_{k,p} \right) v_k$$

Luego,

$$\sum_{n=0}^m \sin nx = \sum_{n=0}^{b_{m-1}} (-1)^n (\sin x)^{2n+1} \left( \sum_{k=2n+1}^m (\cos x)^{a_{n-1}} \left( \sum_{p=0}^n \binom{k}{2p+1} \binom{b_{k-1}-p}{b_{k-1}-n} \right) \right) = \frac{\sin(m+1)\frac{x}{2}}{\sin \frac{x}{2}} \sin m \frac{x}{2}$$

*Observación:*

$$\sum_{n=0}^m u_n \left( \sum_{k=0}^{b_{n-1}} v_k w_{n,k} \right) = \sum_{n=0}^{b_{m-1}} \left( \sum_{k=2n+1}^m u_k w_{k,n} \right) v_n \wedge u_0 = 0$$

Consideremos de (3):

$$\sin(2n+1)x = \sum_{k=0}^n \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \quad (4)$$

Si se hace  $x = \frac{\pi}{2n+1}$ , se obtiene:

$$\sum_{k=0}^n \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k \left( \sin \frac{\pi}{2n+1} \right)^{2k} = 0$$

Si  $n = 2$ , se obtiene la expresión  $16 \left( \sin \frac{\pi}{5} \right)^4 - 20 \left( \sin \frac{\pi}{5} \right)^2 + 5 = 0$ . Puesto que  $\sin \frac{\pi}{6} < \sin \frac{\pi}{5} < \sin \frac{\pi}{4}$ , entonces

$$\sin \frac{\pi}{5} = \frac{1}{2} \sqrt{\frac{5-\sqrt{5}}{2}}.$$

Por otro lado, de (4):

$$\sum_{n=0}^{b_m} \sin(2n+1)x = \sum_{n=0}^{b_m} \left( \sum_{k=n}^{b_m} \left( \sum_{p=0}^n \binom{2k+1}{2p+1} \binom{k-p}{k-n} \right) \right) (-1)^n (\sin x)^{2n+1} = \frac{(\sin b_m x)^2}{\sin x} \quad (5)$$

*Observación:*

$$\sum_{n=0}^m \left( \sum_{k=0}^n u_k v_{n,k} \right) = \sum_{n=0}^m \left( \sum_{k=n}^m v_{k,n} \right) u_n$$

Consideremos de (4):

$$\sin(2n+1)x = (-1)^n (\sin x)^{2n+1} \sum_{p=0}^n \binom{2n+1}{2p+1} + \sum_{k=0}^{n-1} \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1}$$

Por consiguiente,

$$(\sin x)^{2n+1} = \frac{(-1)^n}{2^{2n}} \left( \sin(2n+1)x - \sum_{k=0}^{n-1} \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \right)$$

Observación:

$$\sum_{p=0}^n \binom{2n+1}{2p+1} = 2^{2n}$$

Ahora, usando (5) se obtiene:

$$\begin{aligned} \sum_{m=0}^q (\sin mx)^{2n+1} &= \frac{(-1)^n}{2^{2n}} \left( \frac{\sin(q+1) \frac{(2n+1)x}{2}}{\sin \frac{(2n+1)x}{2}} \sin q \frac{(2n+1)x}{2} - \sum_{m=0}^q \left( \sum_{k=0}^{n-1} \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin mx)^{2k+1} \right) \right) \\ &= \frac{(-1)^n}{2^{2n}} \left( \frac{\sin(q+1) \frac{(2n+1)x}{2}}{\sin \frac{(2n+1)x}{2}} \sin q \frac{(2n+1)x}{2} - \sum_{k=0}^{n-1} (-1)^k \left( \sum_{p=0}^k \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) \left( \sum_{m=0}^q (\sin mx)^{2k+1} \right) \right) \quad (**) \end{aligned}$$

Observación:

$$\sum_{m=0}^q \left( \sum_{k=0}^n u_k v_{m,k} \right) = \sum_{k=0}^n u_k \left( \sum_{m=0}^q v_{m,k} \right) \wedge v_{0,k} = 0$$

Finalmente, hallamos que las expresiones (\*) y (\*\*) son similares a la recurrencia entre coeficientes de la fórmula polinómica general de la suma de las  $n$ -ésimas y  $2n+1$ -ésimas potencias, respectivamente, de los primeros  $q$  naturales.